The Complexity of Models of International Trade

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WINE '09

The Big Picture: Computer Science

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  - Can we compute market equilibria?

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  - What about special models?

The Big Picture: Microeconomics

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  - What determines prices?

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- International Trade:
  - What determines the pattern of trade between countries (agents)?

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  - This should be explained by the model

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- The Heckscher-Ohlin model (1930s):
  - The availability of resources determines trade pattern

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- Problem is back to computing equilibrium

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  - endowment $e_i \in \mathbb{R}^K$ of raw materials
  - production functions

$$f_{ij} : \mathbb{R}^K \to \mathbb{R}$$

The Ricardo Model

- Labor is the only raw material ($K = 1$)
- Production is linear:

$$f_{ij}(x) = a_{ij} \cdot x$$

- a_{ij}: how much of good j country i can make with 1 unit of labor

The Heckscher-Ohlin Model

- Multiple raw materials ($K > 1$)
- Different endowments
- Countries have same production functions:

$$\forall i, i' : \quad f_{ij} = f_{i'j}$$

Complexity Results

- The Ricardo Model

Production	Utilities	Complexity	Note
Linear	Leontief	NP-hard	
	Linear	P	Known[1]
	Leontief	P	"Similar" producers

- The Heckscher-Ohlin Model

Production	Utilities	Complexity	Note
Linear	Leontief	NP-hard	
Linear	Linear	P	
Leontief	Leontief	P	$O(1)$ raw materials

[1] For example, auction algorithm by Kapoor, Mehta, and Vazirani (TCS 2008)

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- Make raw materials tradable, goods non-tradable

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$$u_i(x) = \max u_i(f_{11}(x_{11}) + \cdots + f_{n1}(x_{n1}),$$

$$\ldots, f_{1m}(x_{1m}) + \cdots + f_{nm}(x_{nm}))$$

$$\text{s.t. } \forall i : \quad x_i \geq \sum x_{ij}$$

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- If f_{ij} have constant returns to scale, old and new economies are equivalent!

Ricardo: Leontief Utilities and Similar Producers

Theorem

Let $\mathcal{M}$ be a Ricardo market with Leontief utilities and let $a_i = (a_{i1}, \ldots a_{im}) \in S$ define the production functions of country i. If $|S| = O(1)$ in $\mathcal{M}$ (i.e. there are few distinct a vectors so there is similarity in production), then equilibria are computable in polynomial time.

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Idea:

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- Here:
  - Similar producers $\Rightarrow$ few raw materials and few patterns of trade

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- $O(1)$ distinct vectors a_i in economy implies implies $O(1)$ distinct labor goods

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- m goods $\Rightarrow 2m+1$ distinct ranges for $\frac{w_i}{w_{i'}}$

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- $O(1)$ raw materials $\Rightarrow O(m^{O(1)})$ distinct production patterns!

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Proof: (contd.)

- Given production pattern, upside-down reduction collapses to Leontief exchange economy with $O(1)$ goods
  - Use Devanur and Kannan to compute equilibrium
- Utilities may be arbitrary $\Rightarrow$ production makes equilibrium computation easier!

QED.

n-Country Comparative advantage

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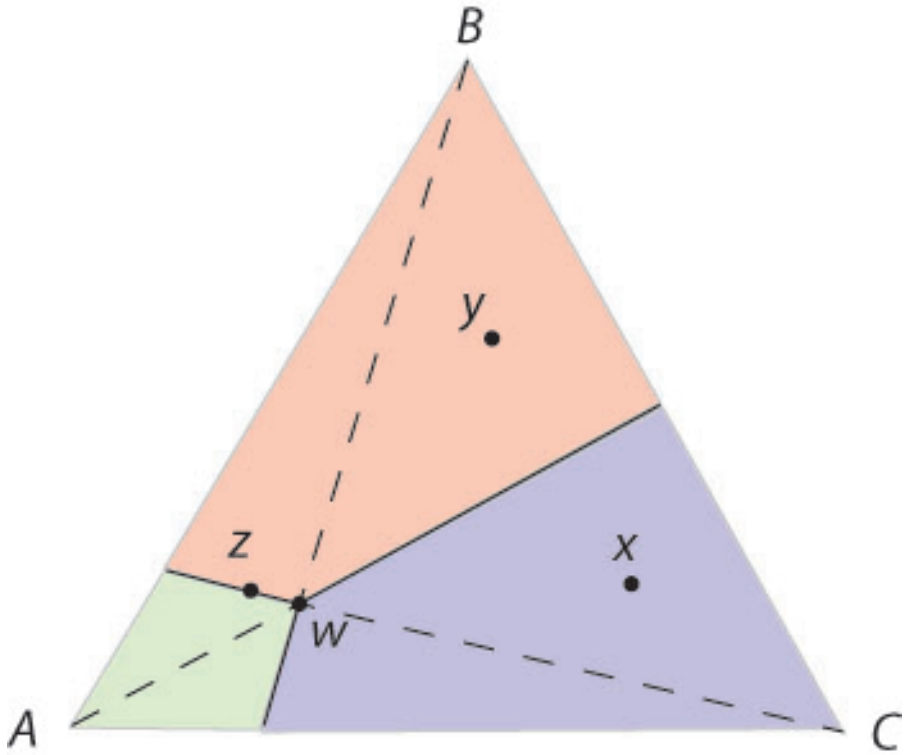
- Map relative production efficiencies (x, y, and z) and prices (w) to $(n-1)$-simplex



- Country A produces goods in green region, etc.

Further Questions

- Reduction with non-constant returns to scale
  - Increasing returns to scale does not work (Papadimitriou and W)
  - "Decreasing returns to scale" should work
- Reduction with more general utilities

- Thank you.