The Complexity of Models of International Trade

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Abstract. We show a range of complexity results for the Ricardo and Heckscher-Ohlin models of international trade (as Arrow-Debreu production markets). For both models, we show three types of results:

1. When utility functions are Leontief and production functions are linear, it is NP-hard to decide if a market has an equilibrium.
2. When utility functions and production functions are linear, equilibria are efficiently computable (which was already known for Ricardo).
3. When utility functions are Leontief, equilibria are still efficiently computable when the diversity of producers and inputs is limited.

Our proofs are based on a general reduction between production and exchange equilibria. One interesting byproduct of our work is a generalization of Ricardo's Law of Comparative Advantage to more than two countries, a fact that does not seem to have been observed in the Economics literature.

1 Introduction

How does production in an economy affect the computability of equilibria? A wave of research has shown a broad spectrum of results for pure exchange economies (e.g. [5, 6, 2]); however, only a handful of papers approach equilibria in the presence of production (e.g. [11, 8, 9]). The papers that do consider production typically construct sophisticated algorithms to compute equilibria, and they do not present negative results.

We take a different approach: in the spirit of Jain and Mahdian's reduction for the Fisher market[7], we reduce production economies to exchange economies. The reduction yields a variety of complexity results for two classical models of trade: the Ricardo model and the Heckscher-Ohlin model. Mathematically, both are special cases of the Arrow-Debreu production market[1]. Economists use them because they represent different motivations for international trade: differentiation in production technology and differentiation in raw materials. For our purposes, their formulations are conveniently simple: the Ricardo model uses linear production functions with a single raw material, and the Heckscher-Ohlin model specifies that agents have identical production functions.

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Our reductions will leverage the plentiful literature on computing equilibria in pure exchange economies. The simplest results reduce the Ricardo and Heckscher-Ohlin models to exchange economies with linear utilities. A wide variety of algorithms already exist for this case — for example, Devanur et al. use a primal dual approach[5] and Garg and Kapoor use an auction algorithm[6].

Our hardness results are based on the NP-hardness result of Codenotti et al. for pure exchange economies[2]. Revisiting their proof yields a convenient tool for showing NP-hardness for our production economies. It is noteworthy that the production model need not be complicated — we show that linear production functions with a single production input suffice to preserve hardness.

Our most interesting computational result is that equilibria may become easier to compute when there are only a few types of producers or raw materials. Devanur and Kannan[4] show that for exchange markets, equilibria in a Leontief exchange economy become easier to compute when there are few goods or agents. We use their result to show that equilibria in the Ricardo model are easy to compute when there are few types of producers. Mathematically, this translates to a type of low-rank constraint on the production coefficients in the Ricardo economy. Similarly, for the Heckscher-Ohlin model, we show that equilibria are efficiently computable under Leontief utilities and production functions independent of the number of goods, provided the number of raw materials is small.

The previous two results are interesting in a broader context because real economies have less variation in technologies and raw materials than they do in consumers, goods, and preferences. For example, typical uses of the Heckscher-Ohlin model[12] employ very few raw materials: labor, land, capital, etc. Thus, our results concern economies which may be closer to reality or to patterns studied by economists.

Our complexity results are summarized in Table 1.

Model	Production	Utilities	Complexity	Note
Ricardo	Linear	Leontief	NP-hard	
		Linear	P	Already known, e.g. [11]
		Leontief	P	Similar producers
Heckscher-Ohlin	Linear	Leontief	NP-hard	
	Linear	Linear	P	
	Leontief	Leontief	P	$O(1)$ raw materials

Table 1. A summary of the results in this paper.

As a bonus, we encounter a novel generalization of a classical theorem of economics: Ricardo's law of comparative advantage. This law states that each of two trading countries will specialize in the production of goods for which its relative labor efficiency is larger, with the ratio of the equilibrium price of labor (wage) as the cut-off point. We establish a multi-dimensional generalization (from the interval $[0, 1]$ to the simplex, see Figure 1).

2 Markets, Equilibria, and Production

We will use four special cases of Arrow and Debreu's market model[1]: the exchange economy, the paring exchange economy, the Ricardo production economy, and the Heckscher-Ohlin production economy.

2.1 Exchange Economies

An exchange economy consists of n agents and m divisible goods. Each agent i is initially given e_{ij} units of good j and has a utility function $u_i(x)$ mapping a bundle of of goods $x = \{x_1, \dots x_m\}$ to a nonnegative utility. Agents trade goods to improve their utilities.

We will use both linear and Leontief utilities in this paper. Linear utility functions take the form

$$u_i(x_1, \dots x_m) = \sum_j \phi_{ij} \cdot x_j \ .$$

A player with Leontief utilities desires goods in fixed proportions. The utility functions take the form

$$u_i(x_1, \dots x_m) = \min_j \frac{x_j}{\phi_{ij}} \ .$$

Let $\Phi = [\phi_{ij}]$ be the matrix of coefficients ϕ_{ij}.

An equilibrium in an exchange economy is an allocation x and a set of prices π such that x maximizes the utility of each agent subject to the budget constraint

$$\sum_j \pi_j \cdot x_{ij} \leq \sum_j \pi_j \cdot e_{ij} \ .$$

The Pairing Leontief Economy. In the pairing model (Ye [13]), agent i is endowed with exactly 1 unit of good i and nothing else. When the agents have Leontief utilities, we call it a pairing Leontief economy. Since endowments are fixed, the pairing Leontief economy is completely specified by Φ.

Codenotti et al. used pairing Leontief economies to show that it is NP-hard to decide whether a general Leontief exchange economy has an equilibrium[2]. In fact, the pairing constraint is not violated by their proof, yielding:

Theorem 1. (Derived from Codenotti et al.[2]) *It is NP-hard to decide whether a pairing Leontief exchange economy has an equilibrium.* (Proof omitted.)

2.2 Production Economies

We will restrict Arrow and Debreu's production model. We say that each agent i has one production function f_{ij} for each *tradable* good j (of m total). Each function $f_{ij}(l)$ maps a bundle l of K *non-tradable* raw materials (indexed by k) to $f_{ij}(l)$ units of the j-th good. An agent is endowed with a bundle of raw materials

l_i (for which he has no utility). For our purposes, the production functions will be either linear or Leontief, parameterized by coefficients a_{ij} with matrix form $A = [a_{ij}]$. As before, each agent has a utility function u_i.

Such a production economy may be understood to operate in two stages. First, agent i chooses a production plan to turn his endowment l_i into a bundle of tradable goods x_i using the functions f_{ij}. Second, the agents exchange goods as in an exchange economy.

We will use w_{ik} to denote the effective price of raw material k of agent i.

The Ricardo Model. The special case with a single raw material ($K = 1$) and linear production technologies was used by economist David Ricardo and is commonly known as the Ricardo model. In this restricted setting, the production functions take the form

$$f_{ij}(l) = a_{ij} \cdot l$$

where l is a scalar. We use l_i to denote the amount of raw material possessed by agent i and w_i the price for agent i's raw material. Historically, the raw material l represents labor and the price w_i represents wages.

The Heckscher-Ohlin Model. The case where there are many inputs but production technologies are identical is known as the Heckscher-Ohlin model. In this model, the form of the production functions is not specified.

3 The Upside-Down Reduction

Many of our theorems reduce a production economy to an *upside-down* exchange economy. In an upside-down economy, trade precedes production — agents trade raw materials, then produce their optimal bundles given the raw materials they acquire. To preserve the possibilities of the original economy, raw materials retain the production technology of their original agent. As a result, the production functions are absorbed into the utilities, as each players' utility function for a bundle of raw materials l will be

$$u_i(l) = \max_{x \in X} u_i(x)$$

where X is the set of all bundles agent i can produce given l. This type of reduction was used by Jain and Mahdian in the context of the Fisher model[7], but we use it more broadly.

When the production functions exhibit constant returns to scale, the production possibilities in the upside-down economy are identical to those in the original production economy. Thus, the equilibria are also identical. We use the fact that both linear and Leontief functions exhibit constant returns to scale.

We denote functions and variables in the upside-down economy with a ($'$). In general, an upside-down economy will have $n' = n$ agents and $m' = (n \times K)$ goods. (Since raw materials carry technology, the raw materials of two agents are different goods.) We index goods as (ik) and use $x'_{(ik)}$ to refer to an amount of raw material k that has the production technology of agent i.

The following lemmas give three cases where the reduction behaves nicely — the Leontief/Leontief, linear/Leontief, and linear/linear cases respectively. The technique is similar, so we only prove the Leontief/Leontief case.

Lemma 2. *When all agents have identical production functions, and both production functions and utilities are Leontief, then*

1. *the utility functions in the upside-down economy are also Leontief with easily computable parameters, and*
2. *we can recover equilibrium prices as*

$$\pi_j = \sum_k \frac{\pi_{(k)}}{a_{jk}} \quad .$$

Proof. Since all agents have identical production functions, there will be K distinct goods in the upside-down economy.

Consider the behavior of a single agent, Alice, and drop her subscripts for clarity. Let $x'_{(k)j}$ be the amount of raw material k that Alice uses to produce good j. We can write the amount of good j that Alice produces as

$$\min_k \frac{x'_{(k)j}}{a_{jk}}$$

and Alice's subsequent utility as

$$u(x') = \min_j \frac{\min_k \frac{x'_{(k)j}}{a_{jk}}}{\phi_j} = \min_k \min_j \frac{x'_{(k)j}}{a_{jk} \cdot \phi_j} \quad .$$

In order to maximize her utility, Alice will distribute each input $x_{(k)}$ over goods so as to maximize $\min_j \frac{x'_{(k)j}}{a_{jk} \cdot \phi_j}$. This will occur when all terms are equal, so we know that

$$\min_j \frac{x'_{(k)j}}{a_{jk} \cdot \phi_j} = \frac{1}{m} \sum_j \frac{x'_{(k)j}}{a_{jk} \cdot \phi_j} = \frac{x'_{(k)}}{m} \sum_j \frac{1}{a_{jk} \cdot \phi_j} \quad .$$

Substituting gives Alice's utility function:

$$u(x') = \min_k \left(\frac{x'_{(k)}}{m} \sum_j \frac{1}{a_{jk} \cdot \phi_j} \right) \quad .$$

As claimed, this is Leontief. Moreover, the coefficients ϕ' may be easily computed from ϕ, a, and m.

Since there is only one production technology for each good, we can compute the price of good j as the total cost of the inputs required to make one unit:

$$\pi_j = \sum_k \frac{\pi_{(k)}}{a_{jk}} \quad .$$

□

Lemma 3. *When there is a single type of raw material, production functions are linear, utilities are Leontief, and for all goods j we are told that agents use the raw material of agent i_j to produce good j, then*

1. *the utility functions in the upside-down economy are Leontief and easily computable, and*
2. *we can recover equilibrium prices as*

$$\pi_j = \frac{\pi_{(i_j)}}{a_{i_j j}} \ .$$

Lemma 4. (Like Jain and Mahdian with multiple raw materials[7].) *When the production functions and utilities in the production economy are linear, then*

1. *the utility functions in the upside-down economy are linear and easily computable, and*
2. *equilibrium prices π in the original economy may be recovered from equilibrium prices π' in the upside-down economy as*

$$\pi_j = \min_{i,k} \frac{\pi'_{(ik)}}{a_{ijk}} \ .$$

4 Computability in the Ricardo Model

We show a broad range of computational results for the Ricardo model. Computability with linear and Leontief utilities parallels the exchange economy. Interestingly, we find that with Leontief utilities, equilibria are efficiently computable when producers are sufficiently similar.

Linear Utilities

As a warm-up, we use an upside-down reduction to show that Ricardo equilibria are efficiently computable when the utility functions are linear. (The computability was already known, e.g. the auction algorithm of Kapoor et al.[11].) Note that Jain and Mahdian use the same proof for the Fisher model in [7].

Theorem 5. *Equilibria in the Ricardo model are efficiently computable when agents' utility functions are linear.*

Proof. The Ricardo model, with linear production functions and one raw material, is a special case of the linear production economy reduced in Lemma 4. Following this lemma, the upside-down counterpart to the linear Ricardo economy has linear utility functions that are efficiently computable from the original utilities. Furthermore, we can recover equilibrium prices from the upside-down equilibrium and use them to compute demands (given prices, it is easy to compute demands under linear utilities.)

To complete the proof, we note that many algorithms exist to compute equilibria in linear exchange economies, e.g. [5, 6]. Thus, linear Ricardo equilibria are efficiently computable. □

Leontief Utilities

While it seems nontrivial to reduce a general exchange economy to a Ricardo economy, it is easy to reduce a pairing one — this yields the following hardness result:

Theorem 6. *It is NP-hard to decide whether a Ricardo model economy with Leontief utility functions has an equilibrium.*

Proof. Let Φ represent the preferences in a pairing Leontief exchange economy. Observe that choosing $A = I$ and $l_i = 1$, i-th country can only produce the i-th good and can produce at most 1 unit of it. Since it has no value for the raw material, it may be assumed to produce 1 unit of the i-th good.

Since each agent i has the same goods for trade and the same utilities in both economies, the equilibria must also be the same. NP-hardness follows from Theorem 1. □

Similar Producers

We show that equilibria are efficiently computable when the utility functions are Leontief provided that the producers are similar. Specifically, we will require a low-rank-like requirement on the matrix of production parameters A.

First, we make the following common observation about the Ricardo model:

Observation 7. *In equilibrium, agent i may produce good j only if $\pi_j = \frac{w_i}{a_{ij}} = \min_i \frac{w_i}{a_{ij}}$. Alternatively, country i may produce a good if and only if $\frac{w_i}{w_{i'}} \leq \frac{a_{ij}}{a_{i'j}}$ for all other countries i'.*

Intuitively, this holds because when $\pi_i < \frac{w_i}{a_{ij}}$, then country i loses by producing good j. On the other hand, if $\pi_i > \frac{w_i}{a_{ij}}$, then a buyer would resist buying and force the price down.

A key insight is that given prices (which may be completely specified by either the π_i's or the w_i's,) the pattern of production is fixed. This will allow us to prove the following lemma decomposing the price space into production patterns:

Lemma 8. *In a Ricardo economy with n producers and m goods, there are at most $O(m^{O(n^2)})$ distinct production patterns. Moreover, each production pattern occurs in a convex polytope in the price space.*

Proof. Observation 7 implies that if

$$\frac{a_{i1}}{a_{i'1}} \geq \dots \geq \frac{a_{ik}}{a_{i'k}} > \frac{w_i}{w_{i'}} > \frac{a_{i(k+1)}}{a_{i'(k+1)}} \geq \dots \geq \frac{a_{im}}{a_{i'm}} \ ,$$

then agent i cannot produce any good for which $\frac{w_i}{w_{i'}} > \frac{a_{ij}}{a_{i'j}}$ while agent i' cannot produce any good for which $\frac{a_{ij}}{a_{i'j}} > \frac{w_i}{w_{i'}}$. Thus, we may give a combinatorial specification of the pattern of production between two countries by specifying where $\frac{w_i}{w_{i'}}$ appears in the ordering of goods. Note that there are $(2m + 1)$ possibilities.

Extending this idea to n agents, we want to show that specifying the pairwise combinatorial production patterns will specify the overall pattern of production. For a given good j, either

1. There is some agent i such that $\frac{a_{ij}}{a_{i'j}} > \frac{w_i}{w_{i'}}$ for all other agents i', or
2. there is a cycle of agents i, i_2,... i_r such that $\frac{a_{ij}}{a_{i_2j}} > \frac{w_i}{w_{i_2}}$, ... $\frac{a_{i_rj}}{a_{ij}} > \frac{w_{i_r}}{w_i}$.

However, option (2) is impossible: multiplying the first $r-1$ inequalities gives $\frac{a_{ij}}{a_{i_rj}} > \frac{w_i}{w_{i_r}}$, which contradicts the final inequality. Thus, for each good, there is some producer who can produce it. It follows that specifying all the combinatorial pairwise patterns must specify the overall pattern of production.

Since there are $O(n^2)$ pairs of agents and $(2m+1)$ patterns for each pair, there are at most $O(m^{O(n^2)})$ different combinatorial characterizations and therefore production patterns in the economy.

Finally, note that the production pattern will be specified by $O(n^2)$ inequalities of the form

$$\frac{a_{ik}}{a_{i'k}} > \frac{w_i}{w_{i'}} > \frac{a_{i(k+1)}}{a_{i'(k+1)}}$$

or an equality of the form

$$\frac{a_{ik}}{a_{i'k}} = \frac{w_i}{w_{i'}}$$

Each equality/inequality bounds the equilibrium prices between a pair of hyperplanes. The union of the hyperplanes defines the convex polytope in the price space in which this production pattern occurs. □

Lemma 9. *If the rows of the production matrix A are scalar multiples of $K = O(1)$ different vectors, then computing eqilibria in the Leontief Ricardo economy is as easy as finding equilibria in a Leontief exchange economy with K goods restricted to a convex polytope in the price space.*

Proof. Briefly, the $K = O(1)$ bound dictates that there are $K = O(1)$ interesting raw materials. Combined with Lemma 8, we will conclude that there are a polynomial number of distinct production patterns. This permits an upside-down reduction for each production pattern using Lemma 3 to reduce to a Leontief exchange economy.

First, Observation 7 and our restriction on A will imply that agents see K distinct producers in the economy. Let A_i denote the i-th row of A. Let i and i' be agents whose production vectors are scalar multiples, i.e. $A_i = c \cdot A_{i'}$ for some constant c. We claim that in equilibrium, $w_i = c \cdot w_{i'}$, and therefore agents are ambivalent between having one unit of i's raw material and c units of i''s raw material.

Assume the contrary, i.e. $w_i \neq c \cdot w_{i'}$ If $w_i < c \cdot w_{i'}$, then $\frac{w_i}{a_{ij}} < \frac{w_i'}{a_{i'j}}$ for all goods j. By Observation 7, this implies that agent i' does not produce anything. Similarly, $w_i > c \cdot w_{i'}$ would imply that agent i does not produce anything. This can only happen in equilibrium if neither agent i nor agent i' produce anything, in which case it must be that $w_i = w_{i'} = 0$.

Thus, the raw materials of i and i' are indistinguishable. It follows that from a computational perspective, we need only consider an economy with K distinct producers (we can normalize so that $A_i = A_{i'}$.)

According to Lemma 8, this implies that there are at most $O(m^{O(K^2)}) = O(m^{O(1)})$ different production patterns. Since we have one raw material, linear production functions, Leontief utilities, and knowledge of the production pattern, we apply Lemma 3 to reduce the problem to a Leontief exchange economy with K goods in a polytope in the price space. (The relationship between prices in the Ricardo and upside-down economies tells us how to transform the Ricardo price polytope to the price space of the upside-down economy.) □

Theorem 10. *If the rows of the production matrix A are scalar multiples of $K = O(1)$ different vectors in a Leontief Ricardo economy, then equilibria are efficiently computable.*

Proof. We use the method of Devanur and Kannan[4] to compute equilibria in a polytope for a Leontief exchange economy. Combining this with Lemma 9 gives a polynomial time algorithm. □

4.1 Ricardian Comparative Advantage

The price-space decomposition implied by Observation 7 suggests a new generalization of Ricardo's law of comparative advantage. A well-known theorem in economics, Ricardo's law of comparative advantage for two agents (originally countries) is as follows:

Theorem 11. *(David Ricardo) In equilibrium for a two agent Ricardo economy, if the goods are ordered by relative production factors a_{ij} and equilibrium raw material prices w_i as*

$$\frac{a_{11}}{a_{21}} \geq \ldots \geq \frac{a_{1k}}{a_{2k}} > \frac{w_1}{w_2} > \frac{a_{1(k+1)}}{a_{2(k+1)}} \geq \ldots \geq \frac{a_{1m}}{a_{2m}}$$

then agent 1 produces goods 1 through k and agent 2 produces goods $(k+1)$ through m. If, for some good j we have $\frac{w_1}{w_2} = \frac{a_{1j}}{a_{2j}}$, then either country may produce good j.

Interestingly, previous attempts to generalize comparative advantage failed to produce a nice theory[10, 3]. However, hyperplane-partitioning leads to the following intuitive generalization:

Theorem 12. *Comparative advantage in an n-agent Ricardo economy may be understood as a partition of an $(n-1)$-dimensional simplex by the price vector w into n convex polytopes. A good j is produced by country i if and only if its relative production technologies map to a point in i's polytope.*

Proof. Observation 7 tells us that in equilibrium, country i produces all goods for which $\frac{w_i}{w_i'} < \frac{a_{ij}}{a_{i'j}}$ for all other countries i', and that it may produce goods for which $\frac{w_i}{w_i'} \leq \frac{a_{ij}}{a_{i'j}}$ for all i'.

Consider the material prices and production coefficients as vectors

$$w = (w_1, \ldots, w_n), \;\; a_j = (a_{1j}, \ldots, a_{nj})$$

and normalize them according to their L_1 norm,

$$w' = \frac{w}{|w|_1}, \;\; a'_j = \frac{a_j}{|a_j|_1} \;\; .$$

Since all a_{ij} are positive, this maps the wage and production vectors to points on the $(n-1)$-dimensional simplex.

The points π in the price space where $\frac{\pi_i}{\pi_{i'}} = \frac{w_i}{w_{i'}}$ form a hyperplane. By Observation 7, this hyperplane partitions the space (and therefore the simplex) between goods possibly produced by i and goods possibly produced by i'. Together, the hyperplanes partition the simplex into n convex polytopes P_i where country i produces those goods whose normalized production technology a'_j falls inside P_i.

Figure 1 illustrates the generalization. □

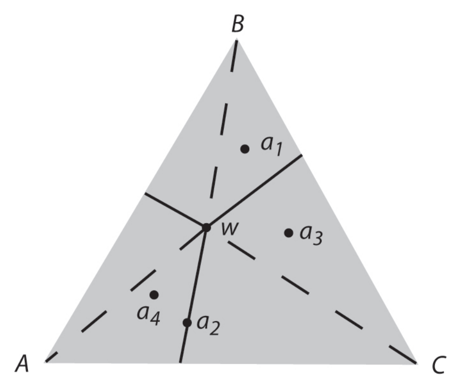

Fig. 1. *Comparative advantage for 3 agents in the Ricardo model.* The space of relative production ratios is visualized as a 2-dimensional simplex (i.e. triangle) following Theorem 12. If w represents the equilibrium price vector for the raw material, then good 1 will be produced by country B, 3 will be produced by C, 4 will be produced by A, and 2 may be produced by either A or C.

5 Computability in the Heckscher-Ohlin Model

The Heckscher-Ohlin model stipulates that agents' production functions are identical. Again, we show a variety of results and, most interestingly, see that when the number of raw materials is small ($K = O(1)$), the number of goods may not matter (see Corollary 15).

Theorem 13. *It is NP-hard to determine if a Heckscher-Ohlin economy with linear production functions and Leontief utilities has an equilibrium.*

Proof. Like Theorem 6, it is easy to simulate a pairing Leontief exchange economy. Let Φ parameterize a paring Leontief exchange economy. Construct a Heckscher-Ohlin economy with n raw materials, n outputs, and production functions parameterized by

$$a_{jk} = \begin{cases} 1 \; i = j \\ 0 \text{ otherwise} \end{cases}$$

Endow agent i with one unit of raw material i and nothing else, i.e.

$$l_{ik} = \begin{cases} 1, \, i = k \\ 0, \text{ otherwise} \end{cases}$$

Agent i can produce exactly one unit of good i and nothing else, so the goods for trade are identical to the pairing Leontief economy. Thus, the equilibria must be the same. □

Our next two results will be corollaries of the following theorem:

Theorem 14. *When the utility and production functions are both linear (or both Leontief), computing equilibria in the Heckscher-Ohlin model reduces to computing equilibria in an exchange economy with linear (Leontief) utilities and K goods.*

Proof. The linear and Leontief cases are straightforward applications of Lemmas 4 and 2 respectively. In both cases, the reductions are efficiently computable. □

Corollary 15. *When the utility and production functions are both Leontief and there are $K = O(1)$ raw materials, equilibria in the Heckscher-Ohlin model are efficiently computable.*

Proof. It is sufficient to compute equilibria in an exchange economy with Leontief utilities and $m = O(1)$ goods. Devanur and Kannan show that such equilibria are efficiently computable[4]. □

Corollary 16. *When the utility and production functions are both linear, equilibria in the Heckscher-Ohlin model are efficiently computable.*

Proof. It is sufficient to compute equilibria in an exchange economy with linear utilities, a problem for which many efficient algorithms exist, e.g. [5, 6]. □

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