Market Communication in Production Economies

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A Communication Bound for Market Equilibrium

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A Communication Bound for Market Equilibrium

- In a market with m goods, m prices are sufficient to communicate an equilibrium...
- *BUT* we show the number of *bits* of information must be polynomial in
    - m,
    - n, the number of agents, and
    - l, the number of production firms.

Communication Complexity

- Yao's 2-party model:
  - Alice has a private, n-bit string a
  - Bob has a private, n-bit string b
  - Alice and Bob want to compute $f(a, b)$
  - How many bits of communication are required to compute $f(a, b)$?

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- *Example:* Alice and Bob have strings a and b. Are a and b different?

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...but market equilibrium doesn't fit this pattern:

1. The "Invisible hand" publishes prices.
2. Agents sell endowment to the market.
3. Firms maximize profit.
4. Agents buy optimal bundle given market prices.
5. The market clears.

Market Communication

Deng, Papadimitriou, and Safra (STOC 2002)

Setting:

- n agents, each agent $i \in [n]$...
  - has private information $x_i \in X_i$, and
  - must compute $f_i(x_1, \ldots x_n)$.
- Agent 0, the "invisible hand," knows $(x_1, \ldots x_n)$.

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Protocol:

1. Agent 0 computes $x_0 = g_0(x_1, \ldots x_n)$
2. Agent 0 broadcasts x_0
3. Each agent i computes $f_i(x_1, \ldots, x_n) = g_i(x_0, x_i)$

Market Communication

Definition

A *market communication* protocol is a set of functions $(g_0(\cdot), g_1(\cdot), \ldots g_n(\cdot))$ where $g_0 : X_1 \times \ldots X_n \rightarrow X_0$, and $g_i(g_0(x_1, \ldots x_n), x_i) = f_i(x_1, \ldots x_n)$.
The amount of market communication is the size of $x_0 = g_0(x_1, \ldots x_n)$.

An Arrow-Debreu Market

- m tradable, divisible goods
- n agents with...
  - utility functions

$$u_i : \mathbb{R}_+^m \to \mathbb{R}$$

  - endowment $e_i \in \mathbb{R}_+^m$

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    $$u_i : \mathbb{R}_+^m \to \mathbb{R}$$
  - endowment $e_i \in \mathbb{R}_+^m$
- l production firms...
  - with a set of production possibilities
    $$Y_k \subset \mathbb{R}^m$$
  - agent i owns a $\sigma_{i,k}$ share of firm k

Market Equilibrium

- Equilibrium is tuple (x, y, π) such that
  - x_i is an optimal consumption bundle:

$$x_i \in \text{argmax}_{x \mid 0 \leq \pi \cdot x \leq M_i} u_i(x)$$

    where $M_i = \pi \cdot e_i + \sum_k \sigma_{i,k} (\pi \cdot y_k)$
  - y_k is an optimal production plan:

$$y_k \in \text{argmax}_{y \in Y_k} \pi \cdot y$$

  - The market clears:

$$0 \leq \sum_i x_i \leq \sum_i e_i + \sum_k y_k$$

Related Work

- Dimensionality of message spaces
  - Economists' analog to communication complexity
  - $m - 1$ real numbers (prices) are optimal for representing a market equilibrium
  - Calsamiglia (1988) — parametric communication (analogous to market communication) does not help.
- Nisan and Segal (2006) — "prices" are required to communicate an allocation
- Deng, Papadimitriou, and Safra (2002) — Market communication

DPS 2002

Theorem (Deng, Papadimitriou, and Safra, STOC 2002)

If the non-satiation requirement for utilities is relaxed, communicating an exchange equilibrium in the market communication model requires

$$\Omega\left(n \log(m+n)\right)$$

bits of communication.

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Weaknesses:

- Relaxes non-satiation requirement.
- Proof uses $\Omega(\frac{n}{m})$-bit numbers in players' endowments.

Why $\Omega(\frac{n}{m})$-bit Endowments Matter

Lemma

If e_{ij} is represented using β bits, then communicating a market equilibrium in the market communication model requires

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$\beta = \Omega(\frac{n}{m})$ implies $\Omega(n)$ bits of communication are necessary

Why $\Omega(\frac{n}{m})$-bit Endowments Matter

Proof Sketch ($m = n = 2$).

- Agent 1...
  - Has 1 units of good A
  - Only wants good B
- Agent 2...
  - Has $e_{2,B}$ unit of good B
  - Only wants good A

□

Why $\Omega(\frac{n}{m})$-bit Endowments Matter

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- Agent 1...
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- Agent 2...
  - Has $e_{2,B}$ unit of good B
  - Only wants good A
- In equilibrium, agent 1 gives all of A, gets all of B
- Agent 1 must learn $e_{2,B}$, i.e. β bits of information.

□

Main Result

Theorem

In the market communication *model of Deng, Papadimitriou, and Safra (STOC 2002), computing a market equilibrium requires*

$$\left(\frac{3}{4}m - 1\right)(\beta + \lg(n-1)) + n + l - O(1)$$

bits of communication. Moreover, if the number of possible utility functions and production sets is exponential in m, then

$$\left(\frac{3}{4}m - 1\right)(\beta + \lg(n-1)) + (m-2)(n+l) - O(m)$$

bits of information are required.

Main Result: Strong Version

Lemma

In the market communication model, for a market where

- $u_i \in \mathbb{U}$,
- $Y_k \in \mathbb{Y}$, *and*
- β *is the number of bits used to represent each* e_{ij},

computing a market equilibrium requires

$$\left(\frac{3}{4}m - 1\right)(\beta + \lg(n-1)) \quad + \quad \left(n - 1 - |\mathbb{U}|^{\frac{2}{m-2}}\right) \lg |\mathbb{U}| \quad + \quad \left(l - 1 - |\mathbb{Y}|^{\frac{4}{m}}\right) \lg |\mathbb{Y}|$$

bits of communication.

$|\mathbb{U}| = |\mathbb{Y}| = 2$ gives the first bound of the theorem, $|\mathbb{U}| = |\mathbb{Y}| = 2^{m-2}$ gives the second bound.

Main Result: Strengths

- Gives a stronger bound than DPS02 without $\Omega(n)$-bit endowments.
- Gives intuition and dependence on market parameters.
- Does not relax non-satiation assumption.
- Includes production.

Main Result: Weaknesses

- Works because real numbers rarely sum to the same value
  - Likely requires exact equilibria

Proof Sketch: Utilities and Consumption

Proof Sketch.

- Agent $i \geq 2$
  - has utility function

$$u_i(x) = x_{i,1} + \sum_{j=2}^{n} 2\sqrt{\lg c_{i,j}}\sqrt{x_{i,j}}$$

    for $c_{i,j} \in C = \{2, 3, \dots\}$ (the first $|C|$ primes)
  - has an endowment $\vec{e}_i$ where $e_{i,j}$ may be any β-bit number ($j \geq 2$).

Proof Sketch: Utilities and Consumption

Proof Sketch (Continued).

- Agent $i = 1$
  - has utility function

$$u_i(x) = x_{i,1} + \sum_{j=2}^{n} 2\sqrt{x_{i,j}}$$

  - has a fixed (known) endowment $\vec{e}_i$.
- Agent 1 has no private information.

Proof Sketch: Utilities and Consumption

Proof Sketch (Continued).

- For $j \geq 2$, equilibrium consumption by agent 1 will be

$$x_{1,j} = \sum_i e_{i,j} \frac{1}{\sum_i \lg c_{i,j}} = \frac{\sum_i e_{i,j}}{\lg \left(\prod_i c_{i,j} \right)}$$

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- $\prod_i c_{i,j}$ is prime factorization, so $\frac{|C|^{n-1}}{|C|!}$ possibile values
- $(n-1)2^\beta$ possible values for $\sum_i e_{i,j}$, so can show

$$(n-1)2^\beta \times \frac{|C|^{n-1}}{|C|!}$$

possible values for $x_{1,j}$.

Proof Sketch: Utilities and Consumption

Proof Sketch (Continued).

- Aggregating over all $j \geq 2$ and noting that $|\mathbb{U}| = |C|^{m-1}$, we have

$$\left((n-1)2^{\beta} \times \frac{|C|^{n-1}}{|C|!}\right)^{m-1}$$
$$\geq \left((n-1)2^{\beta}\right)^{m-1} |\mathbb{U}|^{n-1-|\mathbb{U}|^{\frac{1}{m-1}}}$$

possible values for x_1.

Proof Sketch: Utilities and Consumption

Proof Sketch (Continued).

- Since agent 1 has no private information, there must be at least as many communication sequences as values of $\vec{x}_1$
- Thus,

$$(m-1)(\beta + \lg(n-1)) + \left(n - 1 - |\mathbb{U}|^{\frac{1}{m-1}}\right) \lg |\mathbb{U}|$$

bits of communication are required to reach equilibrium.

Proof Sketch: Production

Proof Sketch (Continued).

- Firm $k \geq 2$ has production function for goods $j \equiv 1 \mod 2$:

$$y_{j,k} = f_{j,k}(y_{1,k}) = 2\sqrt{y_{(j-1),k} \cdot \lg c_{j,k}}$$

- Agents only want goods $j \geq 2$
- Like consumption,

$$\frac{m}{2}(\beta + \lg(n-1)) + \left(n - 1 - |\mathbb{Y}|^{\frac{2}{m}}\right) \lg |\mathbb{Y}|$$

bits of communication are required

Proof Sketch: Finale

Proof Sketch (Continued).

- Putting the two results side-by-side in a single market requires

$$\begin{aligned}\left(\frac{m}{2}-1\right)(\beta+\lg(n-1)) \quad &+ \quad \left(n-1-|\mathbb{U}|^{\frac{2}{m-2}}\right)\lg|\mathbb{U}| \\ \frac{m}{4}(\beta+\lg(n-1)) \quad &+ \quad \left(l-1-|\mathbb{Y}|^{\frac{4}{m}}\right)\lg|\mathbb{Y}|\end{aligned}$$

bits of communication.

□

Main Result, Reprieve

Theorem

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bits of communication. Moreover, if the number of possible utility functions and production sets is exponential in m, then

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bits of information are required.

Conclusion

- Lower bound tarnishes power of prices.
- A result for approximate equilibria could be more meaningful.

- Thank you.